provided that

$$[(J_3 + mhr_1) - (J_1 + mh^2)]\,\omega^2 - mg\,(h - r_1) > 0\ (< 0) \tag{5.4}$$

According to the results in paragraph 4, the inequality (5.4) also defines the stability condition for the rotation of a top on a plane with high viscous friction if the above-mentioned air resistance is taken into account in addition to the friction against the plane. We note that, unlike inequality (5.3), which is valid for any value of the coefficient of friction not equal to zero or infinity, inequality (5.4) is only valid when the value of this coefficient is fairly large. In the general case, the stability of the rotation of a top on a plane with friction allowing for air resistance is determined by a rather cumbersome inequality and depends on the ratio of the coefficients of sliding friction and the air resistance.

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IMPACTS IN A SYSTEM WITH CERTAIN UNILATERAL COUPLINGS*

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The characteristics of the dynamics of a system with ideal unilateral couplings resulting from the possibility of a simultaneous impact against two or more couplings are studied.

It is shown that a correct definition of an impact impulse during repeated impact is only possible in exceptional cases, that is, if the couplings are orthogonal or the impact is of an absolutely inelastic nature (in spite of the elasticity of each coupling individually). In the general case a percussive impulse does not possess the property of a continuous dependence on the initial conditions and the number of surfaces of discontinuity in phase space increases rapidly as the number of repetitions of the impact increases. In view of this, the problem of determining the post-impact motion in systems with a large number of unilateral couplings is of a stochastic nature.

The equations of motion are regularized in the case of orthogonal couplings and absolutely elastic collisions. Examples are considered which show the effect of the geometric and elastic properties of the couplings on the motion of certain mechanical systems.

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1. Let a system of solids be described by the Lagrange function

$$L = T + U, \quad T = {}^1/_2 \mathbf{q}^{\cdot} A(\mathbf{q}) \mathbf{q}^{\cdot T}, \quad U = U(\mathbf{q}), \quad \mathbf{q} \in R^n \tag{1.1}$$

and $k < n$ frictionless couplings $q_j \geqslant 0, \ j = 1, \ldots, k$.

If there are no impacts on the couplings the motion is described by the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{q}^{\cdot}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{R} \tag{1.2}$$

where $R_j = 0$ when $q_j \neq 0$ $(j = 1, \ldots, k)$, $R_{k+1} = \ldots = R_n \equiv 0$.

If, at a certain instant of time, it turns out that $q_\alpha = 0$ $(\alpha \in I \subseteq \{1, \ldots, k\})$ and $q_{\alpha^0}^{\cdot} < 0$ for a certain $\alpha^0 \in I$, there is an impact on the system against the coupling q_α.

In the case when I consists of a single element, the impact can be described directly by means of classical stereomechanical theory /1/ which is based on the hypothesis of a vanishingly small duration of a collision. The generalized coordinates do not change during an impact and the pre- and post-impact values of the generalized velocities are linked by the relationships

$$(\mathbf{q}^{\cdot +}, \mathbf{e}_j)_\mathbf{q} = (\mathbf{q}^{\cdot -}, \mathbf{e}_j)_\mathbf{q}, \quad j \overline{\in} I, \quad q_\alpha^{\cdot +} = -\varkappa_\alpha q_\alpha^{\cdot -}, \quad \alpha \in I \tag{1.3}$$

where $\mathbf{e}_j$ is a row of zeros with a unity in the j-th position and the scalar product $(,)_\mathbf{q}$ is defined with the aid of the kinetic energy matrix of the system:

$$(\mathbf{u}, \mathbf{v})_\mathbf{q} = \mathbf{u} A(\mathbf{q}) \mathbf{v}^T, \quad \mathbf{u}, \mathbf{v} \in R^n \tag{1.4}$$

In the case of a single element set I, Eqs.(1.3) correctly define the values of $\mathbf{q}^{\cdot +}$ in the functions $\mathbf{q}^{\cdot -}$ and $\mathbf{q}$ which also serves as a basis for the adoption of the assumptions of stereomechanics.

Relationships (1.3) can also be used to describe a multiple impact /2/. This definition will, however, be incorrect as the following simple example shows.

Let a uniform sphere strike a dihedral angle of magnitude ${}^2/_3\pi$, moving, before the impact parallel to its bisector plane (Fig.1). In an absolutely elastic impact the sphere, having struck one edge of the angle, rebounds parallel to its other edge. According to the formal definition, it must rebound in the bisector plane.

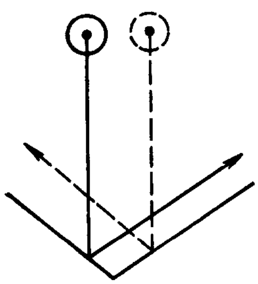

Fig.1

An analysis of relationships (1.3) shows that the first group of them which expresses the absence of friction, also remains valid during a multiple impact /3/ which cannot be said of the second group which express the elastic properties of the couplings. In spite of the fact that, for a single impact, these relationships describe the properties of certain elastic media, in the case of a multiple impact they lose any physical foundation, generally speaking. The reason for this is concealed in the fact that, in the case of non-orthogonal generalized coordinates $\mathbf{q}$, the matrix $A(\mathbf{q})$ is non-diagonal /4/ and the change in the generalized velocity $q_\alpha^{\cdot}$ upon impact depends, by virtue of (1.2), not only on R_α but also on R_β, $\beta \neq \alpha$, $\alpha, \beta \in I$.

In fact, by solving (1.2) for the generalized accelerations, we obtain $\mathbf{q}^{\cdot\cdot} = \mathbf{R}A^{-1} + \ldots$, where the terms which have not been written out do not contain the reactions of the constraints. The independence of $q_\alpha^{\cdot\cdot}$ from R_β when $\alpha \neq \beta$ is expressed by the condition

$$M_{\alpha\beta} = 0 \quad \text{for} \quad q_\alpha = q_\beta = 0, \ \alpha, \beta \in I \tag{1.5}$$

where $M_{\alpha\beta}$ are the elements of the matrix $A^{-1}(\mathbf{q})$.

On the other hand, the vector $\mathbf{N}_j = \mathbf{e}_j A^{-1}$ is orthogonal to the plane $q_j = 0$ in the sense of definition (1.4), since

$$(\mathbf{N}_j, \mathbf{e}_i)_\mathbf{q} = \mathbf{e}_j A^{-1} A \mathbf{e}_i^T = \mathbf{e}_j \mathbf{e}_i^T = \delta_{ij}$$

By noting that

$$(\mathbf{N}_\alpha, \mathbf{N}_\beta)_\mathbf{q} = \mathbf{e}_\alpha A^{-1} \mathbf{e}_\beta^T = M_{\alpha\beta} \tag{1.6}$$

we arrive at the conclusion that relationship (1.5) expresses the orthogonality of the planes $q_\alpha = 0$, $q_\beta = 0$ at their points of intersection.

When conditions (1.5) are satisfied, the multiple impact equations split up: the change in the generalized velocity $q_\alpha^{\cdot}$ only depends on the reaction of the same coupling. When this is so, Eqs.(1.3), which are a generalization of stereomechanical theory to the case of impact against several couplings, correctly determine the impact pulse.

If, however, conditions (1.5) are not satisfied, then, in real physical models, the impact pulse undergoes significant changes when the initial conditions are changed by a magnitude of the order of the elastic deformations.

In order to illustrate this, let us investigate the impact of an absolutely hard sphere against a dihedral angle of magnitude α (Fig.2.a) which is formed by half-spaces with the elastic properties of a Kelvin-Voigt /1/ medium

$$R_i = \eta\varepsilon_i + \mu\varepsilon_i^{\cdot} \quad (i = 1, 2) \tag{1.7}$$

where ε_i is the normal deformation of the corresponding half-space, and R_i is the normal stress (the tangential stresses are equal to zero: frictionless constraints are considered).

The equations of motion of the sphere have the form

$$m\varepsilon_1^{\cdot\cdot} = -R_1 + R_2 \cos\alpha, \quad m\varepsilon_2^{\cdot\cdot} = R_1 \cos\alpha - R_2 \tag{1.8}$$

where the force R_i is equal to zero when $\varepsilon_i \leqslant 0$ and, when $\varepsilon_i > 0$, it is determined by formula (1.7).

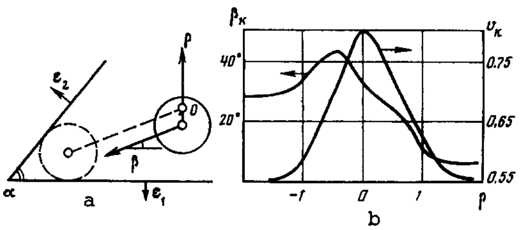

Fig.2

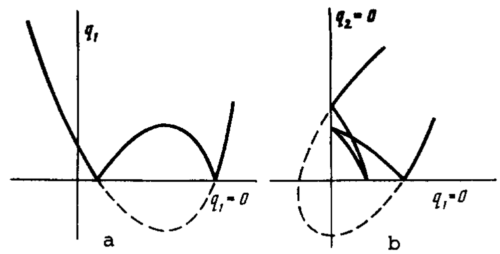

Fig.3

When $\cos\alpha \neq 0$ (the planes are non-orthogonal), a dependence of the intersection of the generalized accelerations on reactions of different kinds, which has been noted above, appears Fig.2b shows the results of a numerical integration of system (1.8) for the values $m = 1, \alpha = 50°, \beta = 10°, \eta = 1, \mu = 0{,}22$ (which corresponds to a value for the coefficient of restitution of $\varkappa = 0.7$ and to an initial velocity of the centre of the sphere $v = 1$ (in the case of a simple single impact, the percussive deformation is approximately equal to unity). The initial displacement of the centre of the sphere from the position in which it is simultaneously in contact with both edges of the angle (Fig.2a) is plotted along the abscissa while the magnitude of the angle of reflection β_k and the modulus of the velocity of the sphere at the end of the impact, i.e. at the moment when $\varepsilon_{1,2} \leqslant 0, \varepsilon_{1,2}^{\cdot} < 0$, are plotted along the ordinate.

It follows from these results that the values $\beta_k = 10°, v_k = 0.5$ which are predicted by system (1.3) are not realized under any initial conditions. Moreover, if the deformations due to the impact are considered to be negligibly small, $|\rho| \gg 1$, the shock pulse has two different values depending on the sign of ρ.

2. Let relationships (1.5) be satisfied for $I = \{1, \ldots, k\}$, i.e. all the unilateral constraints are mutually orthogonal and, furthermore, let $\varkappa_\alpha = 1, \alpha \in I$.

As has been shown in /5/, Eqs.(1.2) and (1.3) can be regularized in the case when $k = 1$, $\varkappa = 1$: they can be replaced by the Lagrange equations for a certain auxiliary system which is free from constraints and has continuous phase trajectories. In order to do this, it is first necessary to make the substitution of the generalized coordinates $q \to Q$ in order that the plane $q_1 = 0$ should transform into $Q_1 = 0$ and the vectors $\mathbf{N}_1 = \mathbf{e}_1 A^{-1}(\mathbf{Q})$ and $\mathbf{e}_1$ when $Q_1 = 0$ should be found to be collinear. After this an auxiliary system is defined in the phase space R^{2n} (without the constraint $Q_1 \geqslant 0$) by a Lagrange function of the form

$$L^*(\mathbf{Q}, \mathbf{Q}^{\cdot}) = L(|Q_1|, Q_2, \ldots, Q_n, \mathbf{Q}^{\cdot})$$

The trajectories of the initial system in configurational space can be obtained from the trajectories of the auxiliary system by means of a mirror reflection in the $Q_1 = 0$ plane of that part of them for which $Q_1 < 0$ (Fig.3a).

An analogous construction is also applicable in the case being considered of $k > 1$ mutually orthogonal constraints. Here, it is also first necessary to make a replacement of the variables such that the planes $q_\alpha = 0$ transform into $Q_\alpha = 0$ and the vectors $\mathbf{N}_\alpha$ when $Q_j = 0$ $(j = 1, \ldots, k)$ are collinear with $\mathbf{e}_\alpha$ (the existence of such a substitution is obvious from geometrical considerations).

The auxiliary system is defined in the phase space R^{2n} by the Lagrangian

$$L^*(\mathbf{Q}, \mathbf{Q}^{\cdot}) = L(|Q_1|, \ldots, |Q_k|, Q_{k+1}, \ldots, Q_n, \mathbf{Q}^{\cdot}) \tag{2.1}$$

The link between the trajectories of the two systems is established by means of a certain number of mirror reflections in the coordinate planes (Fig.3b for $k = 2$). When this is done, the smoothness of the auxiliary trajectories automatically implies that the impact Eqs.(1.3) are satisfied.

In order to illustrate this method let us consider the motion of a material point of unit mass and weight in a vertical plane on or above a certain piecewise-smooth curve $y = f(x)$ where the y-axis is vertical. If $f \in C_1$, then, by putting $q_1 = y - f(x)$, we obtain a system with a single unilateral constraint which has been investigated in /5/. If the curve has angular points such as, for example

$$f(x) = \begin{cases} f_1(x), & x \leqslant x_0 \\ f_2(x), & x \geqslant x_0 \end{cases}$$
$$f_1(x_0) = f_2(x_0), \quad f_1'(x_0) < f_2'(x_0), \quad f_{1,2} \in C_1$$

the substitution $q_1 = y - f(x)$ in (1.1) leads to a discontinuous kinetic energy. Nevertheless, this system can be written in the form of (1.1) but now with two unilateral constraints by putting, in this case, $q_1 = y - f_1(x)$, $q_2 = y - f_2(x)$. The Lagrangian (1.1) has the form

$$L = {}^1/_2 (x^{\cdot 2} + y^{\cdot 2}) - y = {}^1/_2 (f_2' - f_1')^{-2} [q_1^{\cdot 2} (1 + f_2'^2) - 2 q_1^{\cdot} q_2^{\cdot} (1 + f_1'f_2') + (1 + f_1'^2) q_2^{\cdot 2}] - (q_1 + f_1) \tag{2.2}$$

The angle α between the lines $q_1 = 0$ and $q_2 = 0$ is defined by the relationship (since α makes the angle between N_1 and N_2 up to π)

$$\cos\alpha = \frac{-(\mathbf{N}_1, \mathbf{N}_2)}{|\mathbf{N}_1||\mathbf{N}_2|} = \frac{-M_{12}}{(M_{11}M_{22})^{1/2}} = \frac{-(1 + f_1'f_2')}{(1 + f_1'^2)^{1/2}(1 + f_2'^2)^{1/2}}$$

i.e. α is identical to the linear angle between the curves $y = f_1(x)$ and $y = f_2(x)$.

Condition (1.5) appears as

$$f_1(x_0) f_2(x_0) = -1 \tag{2.3}$$

Since the values of $f_1(x)$ when $x > x_0$ and $f_2(x)$ when $x < x_0$ have no effect on the motion of the point, it is possible, when condition (2.5) is satisfied, to redefine them in such a way that it turns out that $f_1'f_2' \equiv -1$, $f_{1,2} \in C_1$. The Lagrange function of the auxiliary system has the form

$$L = {}^1/_2 (f_2' - f_1')^{-2} [q_1^{\cdot 2} (1 + f_2'^2) + (1 + f_1'^2) q_2^{\cdot 2}] - |q_1| - f_1 \tag{2.4}$$
$$f_{1,2} = f_{1,2}(x), \quad f_2(x) - f_1(x) = |q_1| - |q_2|$$

The equations of motion of system (2.4) are written, when $q_1 \neq 0, q_2 \neq 0$, in the form

$$q_1^{\cdot\cdot} + \operatorname{sgn} q_1 (1 + f_1'' x^{\cdot 2}) = 0, \quad q_2^{\cdot\cdot} + \operatorname{sgn} q_2 (1 + f_2'' x^{\cdot 2}) = 0 \tag{2.5}$$
$$x^{\cdot} = f_2' (1 + f_2'^2)^{-1} (q_1^{\cdot} \operatorname{sgn} q_1 - q_2^{\cdot} \operatorname{sgn} q_2)$$

If the functions f_1 and f_2 are linear and $f_1 = -a^{-1}(x - x_0)$, $f_2 = a(x - x_0)$, Eqs.(2.5) split up and are rapidly integrated. The change in each of the coordinates is periodic and the magnitude of the periods τ_i depends on the initial conditions, $\tau_i = 4(2E_i)^{1/2}$, $E_i = {}^1/_2 q_i^{\cdot 02} + |q_i^0|$ $(i = 1, 2)$. The trajectories of the auxiliary system in configurational space are analogous to Lissajous figures: if τ_1 and τ_2 are comparable, these trajectories are closed. Otherwise, they are everywhere dense in the rectangle $[-E_1, E_1] \times [-E_2, E_2]$.

In the general case Eqs.(2.5) admit of τ-periodic solutions for which $x \equiv x_0$, $q_1 = q_2 = {}^1/_2 t ({}^1/_2 \tau - |t|)$ when $|t| < {}^1/_2 \tau$ (Fig.4a). Let us investigate, to a first approximation, the stability of these particular solutions with respect to the variables $\xi = x - x_0$, $\xi^{\cdot}$.

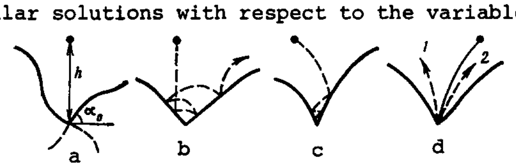

Fig.4

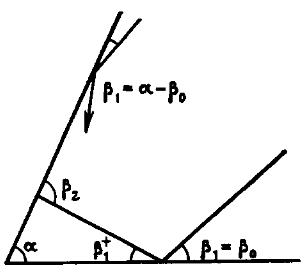

Fig.5

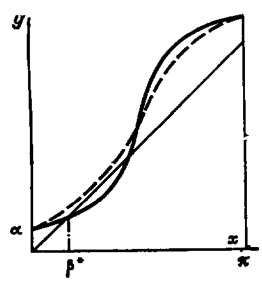

Fig.6

In comparing the equations in variations, one should take account of their different form when $q_1q_2 > 0$ and $q_1q_2 < 0$: in the first case the magnitude of $x^{\cdot 2}$ is of the second order of smallness while, in the second case (motion in the interval between collisions with the first and the second curves, if these moments are not identical), the motion occurs during an interval of time Δt of the first order of smallness. Moreover, the values of the second derivatives in (2.5) are defined differently when $|q_1| > |q_2|$ and $|q_1| < |q_2|$.

For the fundamental matrix of the solutions, one has the expression

$$X(\tau) = \begin{Vmatrix} -1 & -{}^1/_2\tau \\ 2D & -1+\tau D \end{Vmatrix}, \quad D = \frac{a\tau}{4(a^2+1)}\left[af_1''(x_0-0) + \frac{f_2''(x_0+0)}{a}\right] + 4\tau^{-1}, \quad a = f_2'(x_0)$$

The necessary stability condition, $0 < \tau D < 4$ has the form

$$-1 < 2h(\varkappa_1 \operatorname{cosec} \alpha^\circ + \varkappa_2 \sec \alpha^\circ) < 0, \quad \operatorname{tg} \alpha^\circ = a$$

where h is the height of the upward jump of the point in the periodic motion investigated (Fig.4a) and $\varkappa_1$ and $\varkappa_2$ are the curvatures of the curves f_1 and f_2 at the point x_0.

We note that, in the previously discussed case $f_1'' = f_2'' \equiv 0$, the solutions $q_1 = q_2$ are unstable in view of the nature of the trajectories which has been described. Furthermore, if the total energy of the point is sufficiently large, the point will jump out of the well formed by the two perpendicular lines (Fig.4b) with a probability of unity.

3. Now, suppose the orthogonality conditions (1.5) are not satisfied. Since the set of initial conditions which correspond to a multiple impact has a zero measure in phase space, it is of practical interest to study the trajectories lying close to this set. In the case of such trajectories a multiple impact is replaced by simple repeated collisions, during the investigation of which we shall neglect both the duration of each of them as well as the interval of time between them during a single multiple impact. We shall also assume that the coordinates $q_j\ (j \overline{\in} I)$ are orthogonal to $q_\alpha\ (\alpha \in I)$ as a consequence of which $q_j^\cdot$ do not change during impact and it is sufficient to confine ourselves to a consideration of q_α.

Let $I = \{1, 2\}$, $\varkappa_1 = \varkappa_2 = \varkappa$ and let the angle between the planes $q_1 = 0$ and $q_2 = 0$ be equal to α. In an impact against one of the couplings the angles of incidence and reflection are linked by the relationship $\operatorname{ctg}\beta_1^+ = \varkappa^{-1}\operatorname{ctg}\beta_1$ and the angle of incidence upon the next impact against a second coupling is equal to $\beta_2 = \beta_1^+ + \alpha$. The recurrence formula

$$\beta_{m+1} = \alpha + F(\beta_m), \quad F(x) = \operatorname{arcctg}(\varkappa^{-1}\operatorname{ctg} x) \tag{3.1}$$

holds for the subsequent collisions.

Repeated collisions cease after the p-collision if the value of β_{p+1}, defined in accordance with (3.1), turns out to be greater than π. Two versions of the possible mutual positionings of the graphs of $y = \alpha + F(x)$ and $y = x$ are shown in Fig.6.

In the first of the two cases, shown by the broken line, the trajectory, after a finite number of repetitive impacts, leaves the multiple impact zone. In order to determine the percussive pulse it is necessary in this case, in addition to the value of β_0, also to specify the number of the coupling against which the first collision occurs: if this is the first constraint, then $\beta_1 = \beta_0$ and, if it is the second, then $\beta_1 = \alpha - \beta_0$ (Fig.5).

Calculations using formula (3.1), carried out using the data for the example cited in paragraph 1: $\alpha = 50°$, $\beta = 10°$, $\varkappa = 0.7$, lead to the following values: when the first impact is against the first oopling $\beta_k = 28.3°$ (four repetitive impacts $1+2+1+2$) but, when it is against the second coupling, $-\beta_k = 6.7°$ (three repetitive impacts $2+1+2$) which corresponds to the values of the unilateral limits in Fig.2b.

In the second case shown in Fig.6 by the solid line, the sequence β_m, for any $\beta_0 \in]0, \alpha[$, has an upper limit of β^*, the asymptotically stable root of the equation $\alpha + F(x) = x$. The impact is then analogous to a quasiplastic impact /6/ but differs from it in the fact that total extinction of the velocities $q_1^\cdot$ and $q_2^\cdot$ occurs after an infinitely short time. In this case a multiple impact is therefore similar to an absolutely inelastic impact in spite of the elastic nature of each of the constraints individually.

In order to elucidate to which of these two types a multiple impact belongs, it is necessary to determine the minimum of the function $y = \alpha + F(x) - x$ in the interval $[0, \pi]$. Since $y' = \varkappa(\varkappa^2\sin^2 x + \cos^2 x)^{-1} - 1$, this minimum is attained provided that $\cos^2 x + \varkappa^2 \sin^2 x = \varkappa$ and the case of an arresting impact is realized when the inequality

$$2\operatorname{tg}\alpha \leqslant \varkappa^{-1/2}(1-\varkappa) \tag{3.2}$$

is satisfied.

The fact that conditions (3.2) are satisfied in the case of the example cited in paragraph 2 means that the angular point is an unusual trap. Upon falling within the neighbourhood of this point the moving point "adheres" to the vertex of the angle (Fig.4c). If, however, condition (3.2) is not satisfied the point rebounds from the vertex of the angle along one of two directions depending on the sequence of repetitive collisions (Fig.4d).

It may be shown in a similar way that only two situations are also possible when $\varkappa_1 \neq \varkappa_2$:

either the number of repetitive impacts is bounded for any initial conditions or a multiple impact is of an arresting nature for any initial conditions.

Hence, two types of double impact exist. The first of these involves orthogonal constraints and an arresting impact is characterized by the possibility of correctly defining an impact pulse by means of conditions (1.3) (where, in the case of an arresting impact, it is necessary to assume that $\varkappa_j = 0$). The second type is distinguished by a double value of the impact pulse depending on the geometry of the initial conditions with regard to the surface of bifurcation which corresponds to the point being incident precisely at the vertex of the angle. (This does not exclude values of the impact pulse from being identical for certain values of α and $\varkappa$. For example, the case $\alpha = \pi/m, m \in Z, \varkappa = 1$ also belongs to the first type).

It is considerably more difficult to describe an impact against three or more couplings. The difficulty lies in the fact that if the parameters permit several repetitive collisions, surfaces of bifurcation exist in the region of the initial conditions which correspond to different variations of their alternation. For instance, when $k = 3$ and there are four repeated collisions, versions $1 + 3 + 2 + 3$, $1 + 2 + 1 + 3$, etc. are possible, in all a total of 24 versions. When the dimensionality increases, the number of different versions (when $k = 5$ and there are ten repeated impacts, there are more than a million of them) and, correspondingly, the number of possible values of the impact momentum transforms the problem of determining the latter, in principle, into a stochastic problem (in the sense of /7/): in order to determine the motion of a system after impact it is necessary to known the initial conditions with a practically unattainable accuracy. Such cases are not considered in this paper. Concurrent with this, the above-mentioned cases of the correct determination of the impact pulse: orthogonal constraints and an arresting impact, remain possible for any dimensionality.

4. Let us study the effect of double impacts on a system of two heavy material points suspended on ideal threads, which act as unilateral couplings, and fastened by a wightless rod (Fig.7).

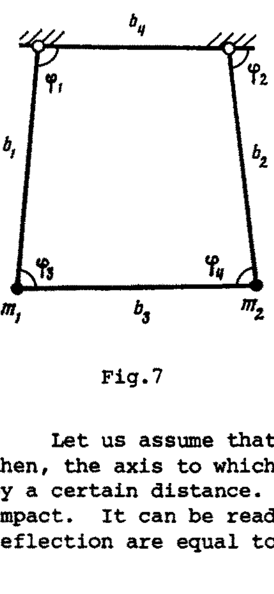

Fig.7

Let r_i, φ_i $(i = 1, 2)$ be the polar coordinates of a point m_i with respect to a system in which the pole is located at the point where the corresponding thread is fastened and the polar axis is horizontal. The system has three degrees of freedom and the Lagrangian (1.1) is written in the form

$$\sum_{i=1}^{2} m_i \left[{}^1/_2 (r_i^{\cdot 2} + r_i^2 \varphi_i^{\cdot 2}) - g r_i \sin \varphi_i\right], \qquad q_i = b_i - r_i \geqslant 0 \tag{4.1}$$

where $\varphi_2, \varphi_2^{\cdot}$ have been eliminated with the help of the relationships

$$(r_1 \cos \varphi_1 - r_2 \cos \varphi_2 - b_4)^2 + (r_1 \sin \varphi_1 - r_2 \sin \varphi_2)^2 = b_3^2 \tag{4.2}$$

$$r_1^{\cdot} \cos \varphi_3 + r_2^{\cdot} \cos \varphi_4 + b_1 \varphi_1^{\cdot} \sin \varphi_3 - b_2 \varphi_2^{\cdot} \sin \varphi_4 = 0$$

System (4.1), (4.2) has a position of stable equilibrium which is determined from the conditions /8/

$$\partial U/\partial q_1 \leqslant 0,\ \partial U/\partial q_2 \leqslant 0,\ \partial U/\partial \varphi_1 = 0$$

whence we obtain

$$m_1 \cos \varphi_1 \sin \varphi_4 + m_2 \cos \varphi_2 \sin \varphi_3 = 0, \quad q_1 = q_2 = 0 \tag{4.3}$$

Let us assume that, up to a certain instant of time, the system is in equlibrium and, then, the axis to which the threads are fastened is displaced downwards in an abrupt manner by a certain distance. This leads to a slackening of the threads and a subsequent double impact. It can be readily seen that, if upon such an impact the angles of incidence and reflection are equal to one another, i.e.

$$q_1^{\cdot +}/q_2^{\cdot +} = q_1^{\cdot -}/q_2^{\cdot -} \tag{4.4}$$

the horizontal component of the impact momentum and its moment with respect to the centre of inertia of the system are equal to zero when condition (4.3) is satisfied. When this is so, the points will move along verticals. If the elastic properties of the threads are identical ($\varkappa_1 = \varkappa_2 = \varkappa$), such a situation arises when the orthogonality condition (1.5) or the arresting impact condition (3.2) is satisfied. Under these conditions the angle α is defined by virtue of (1.6) as:

$$\cos \alpha = M_{12} (M_{11} M_{22})^{-1/2} = \cos \varphi_3 \cos \varphi_4 \left[(\sin^2 \varphi_3 + m_1/m_2)(\sin^2 \varphi_4 + m_2/m_1)\right]^{-1/2}$$

Consequently, the orthogonality condition means that, when the threads are tight, just one of them is perpendicular to the rod.

In the general situation of the bifurcation of the impact momentum a double impact leads to swinging of the rod since condition (4.4) is not satisfied for just one of the two branches. Hence, the orthogonality of the couplings turns out to be a stabilizing factor in the problem under consideration.

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INVESTIGATION OF THE OSCILLATIONS OF ESSENTIALLY NON-LINEAR SYSTEMS WITH INTERNAL RESONANCE*

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Oscillations in systems which do not become linear when the small parameter becomes equal to zero are studied. It is assumed that the generating system contains odd-order resonances. Conditionally periodic solutions of the generating and complete systems are constructed with an accuracy of up to first order in the small parameter. The results obtained represent a further development of the theory of bifurcation of the growth of a cycle from a position of equilibrium.

1. Let us consider an essentially non-linear quasi-autonomous system of $2n$-th order differential equations

$$u_k^{\cdot} = i\nu_k u_k + A_k v^p / v_k + \sum_{l \geqslant 1} \mu^l U_{kl}(u, v, t) \tag{1.1}$$

$$v_k^{\cdot} = \bar{u}_k^{\cdot}; \quad v_k = \bar{u}_k, \quad v^p = v_1^{p_1} v_2^{p_2} \ldots v_n^{p_n}, \quad A_k = \text{const}$$

where μ is a small parameter. The functions U_{kl} are polynomials in u_k, v_k $(k = 1, \ldots, n)$ of an arbitrarily large degree, vanishing when $u = v = 0$, with coefficients conditionally t-periodic and represented by a generalized finite Fourier series. The series in the parameter μ are absolutely convergent when its values are sufficiently small, and the point $u = v = 0$ is a unique singularity in the domain of variation of u and v in question.

We assume that the frequencies are connected by an odd-order resonance relation

$$p_1\nu_1 + \ldots + p_n\nu_n = 0$$
$$(p_i > 0 \ (i = 1, \ldots, n), \ p = \Sigma p_i = 2m + 1 \ (m = 1, 2, \ldots))$$

We note that when we have the internal odd-order resonance and no resonance relations of the same order connecting the eigenfrequencies with the frequencies of the conditionally periodic coefficients, we can reduce, to system (1.1), the arbitrary system of equations of perturbed motion with n pairs of the purely imaginary roots of the form

$$x_k^{\cdot} = -\nu_k y_k + X_k^{(p-1)} + X_k^{(p)} + \ldots, \quad y_k^{\cdot} = \nu_k x_k + Y_k^{(p-1)} + Y_k^{(p)} + \ldots$$

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